NONCOOPERATIVE OLIGOPOLY IN ECONOMIES WITH INFINITELY MANY COMMODITIES AND TRADERS

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In this paper, we extend the noncooperative analysis of multilateral oligopoly to exchange economies with infinitely many commodities and trader types where exchange is modelled using a strategic market game with commodity money and trading posts. We prove the existence of an "active" Cournot-Nash equilibrium and its convergence to a Walras equilibrium when the economy is replicated.

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1. INTRODUCTION

In a multilateral oligopoly model, each trader owns the numeraire good and only one other commodity. Shubik (1973) opened the lined of research on strategic market games by analysing noncooperative exchange in the multilateral oligopoly model. Subsequent research on noncooperative oligopoly in a general equilibrium setting, following Shubik (1973), continues to focus on economies with a finite number of commodities. In this paper, in a multilateral oligopoly strategic market game with commodity money and trading posts (Dubey and Shubik (1978)), we extend the analysis of noncooperative oligopoly to exchange economies with a countably infinite number of commodities and trader types.

The model analysed by Dubey and Shubik (1978) belongs to the line of research initiated by the seminal papers of Shubik (1973), Shapley (1976), and Shapley and Shubik (1977). In this class of models there is a trading post for each commodity where that commodity is exchanged for commodity money. The actions available to a trader are bids, amounts of commodity money given in exchange for other commodities, and offers, amounts of the commodities in trader's initial endowment put up in exchange for commodity money. Prices are determined as ratios between the sum of bids and the sum of offers in each trading post and, if the latter quantity is zero, the price is set equal to zero. Dubey and Shapley (1994) and Codognato and Ghosal (2000) extended strategic market games to exchange economies with a finite number of commodities and a continuum of traders. To the best of our knowledge, no previous attempt has been made to extend such models to economies with an infinity of commodities and traders. In contrast, the cooperative game theoretic approach to modelling oligopoly in exchange economies was extended to economies with a continuum of commodities, by Gabszewicz (1968), and to economies with a countable infinity of commodities by Peleg and Yaari (1970).

Our contributions are as follows. We begin by specifying a well-defined exchange economy with a countable infinity of commodities and trader types with the structure of a multilateral oligopoly. Our approach relies on the literature on economies with infinitely many commodities and with a double infinity of commodities and traders considered by Bewley (1972) and Wilson (1981) respectively. Next, we reformulate the strategic market

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game analysed by Dubey and Shubik (1978) and study the classical problems of existence of a Cournot-Nash equilibrium and convergence to a Walras equilibrium, as the previous contributions in this literature (see also Amir, Sahi, Shubik, and Yao (1990), and Sahi and Yao (1989)).

In a strategic market game, given the price formation and allocation rules, it is straightforward to note that a "trivial" Cournot-Nash equilibrium at which there is no trade and all prices are zero always exists. For this reason, Dubey and Shubik (1978) proved the existence of an "equilibrium point" at which the prices are positive. Subsequently, Cordella and Gabszewicz (1998) showed that it is possible to have an equilibrium point without trade and Busetto and Codognato (2006) found that the role of the positive prices at an equilibrium point without trade is unclear.[1] We address this issue in our setting by adapting the analysis of Bloch and Ferrer (2001) (carried out for the case of a bilateral oligopoly), and we prove the existence of an "active" Cournot-Nash equilibrium at which all commodities are exchanged. Our existence result requires us to solve new technical problems as, in our analysis, the dimension of the commodity space is countably infinite. In addition to the classical assumptions on initial endowments and utility functions (Assumptions 1–4), we impose stronger restrictions on the marginal rate of substitution between the commodity in the trader's initial endowment and commodity money (Assumption 5) which allow us to derive uniform lower and upper bounds on prices, a key step to prove the existence theorem. To obtain this fundamental result, we could not follow the proof strategy based on the Uniform Monotonicity Lemma as in Dubey and Shubik (1978). We then develop a new approach that uses the Generalised Kuhn-Tucker Theorem (see Luenberger (1969)) to characterise traders' best responses. Further, our proof of the existence of an active Cournot-Nash equilibrium relies on an additional assumption on traders' marginal utilities (Assumption 6, a generalisation of Bloch and Ferrer (2001)'s conditions to a framework with more than two commodities). We clarify the role played by Assumption 6 in Example 1. It is worth noting that Assumption 6, differently from Assumption 5, would imply the existence of a Cournot-Nash equilibrium with trade in all trading posts even in a multilateral oligopoly model with a finite number of commodities.

Under our assumptions the existence of a Walras equilibrium is a straightforward consequence of the existence result in Wilson (1981). We show that an active Cournot-Nash equilibrium converges to the Walras equilibrium, when the underlying exchange economy is replicated. Our proof follows the strategy adopted by Dubey and Shubik (1978).

The remainder of the paper is organized as follows. In Section 2 we introduce the exchange economy and the strategic market game. In Section 3 we prove the theorem of existence of an active type-symmetric Cournot-Nash equilibrium. We also provide an example of an economy satisfying all the assumptions we made. In Section 4 we show that the existence of a Walras equilibrium follows from Wilson (1981). In Section 5 we prove the convergence theorem. In Section 6 we discuss our results and we draw some conclusions.

2. THE MODEL

In this section, we specify a multilateral oligopoly model with a countable infinity of commodities and traders and the strategic market game associated to it. We formally state

[1]The formal definition of equilibrium point can be found in Dubey and Shubik (1978). Shapley (1976) introduced the notion of "virtual prices" to provide an economic rationale to the positive price associated to a trading post without trade. The proof of existence of a Cournot-Nash equilibrium having positive virtual prices for the commodities which are not exchanged is an open problem. See Cordella and Gabszewicz (1998) and Busetto and Codognato (2006) for a more detailed analysis.

and discuss the assumptions required to prove the existence of a Walras equilibrium and a Cournot-Nash equilibrium where all commodities are exchanged.

Let T_t be the set of traders of type t and let $k \geq 2$ be the cardinality of each set T_t, for $t = 1, 2, \ldots$. The set of traders $I = \cup_{t=1}^{\infty} T_t$ is the union of all sets of types of trader. The set of commodities is $J = \{0, 1, 2, \ldots\}$ and the consumption set is denoted by X. An element of the set X is a commodity bundle x and the coordinate x_j represents the amount of commodity j in the commodity bundle x. A trader i is characterised by an initial endowment, w^i, and a utility function, $u^i : X \to \mathbb{R}$, which describes his preferences. Traders of the same type have the same initial endowment and utility function. The context should clarify whether the superscript refers to a trader type or to a trader. An exchange economy is a set ${}_k\mathcal{E} = \{(u^i(\cdot), w^i) : i \in I\}$ and the subscript k denotes the number of traders of each type. Let $w_j = \sum_{t=1}^{\infty} w_j^t$ and w be the vector whose coordinates are w_j, for each $j \in J$. In an exchange economy ${}_k\mathcal{E}$ the vector of aggregate initial endowment is then kw. An allocation $\mathbf{x}$ is a specification of a commodity bundle x^i, for each $i \in I$, such that $\sum_{i \in I} x_j^i = kw_j$, for each $j \in J$. Let p be a price vector whose coordinate p_j is the price of commodity j, for each $j \in J$. The budget set of trader i is $B^i(p) = \{x \in X : \sum_{j=0}^{\infty} p_j x_j \leq \sum_{j=0}^{\infty} p_j w_j^i\}$, for any p. A Walras equilibrium is a pair $(p, \mathbf{x})$ consisting of a price vector p and an allocation $\mathbf{x}$ such that x^i is maximal with respect to $u^i(\cdot)$ in i's budget set $B^i(p)$, for each $i \in I$.

We make the following assumptions.

Assumption 1 The consumption set X is the space of non-negative bounded sequences ℓ_∞^+ and it is endowed with the product topology.[2]

Assumption 2 The vector w belongs to ℓ_∞^+.

Assumption 3 The initial endowment of a type t trader is such that $w_0^t > 0$, $w_t^t \geq W$, with W a positive constant, and $w_j^t = 0$, for each $j \in J \setminus \{0, t\}$, for $t = 1, 2, \ldots$.

Assumption 4 The utility function of a type t trader is continuous, continuously Fréchet differentiable, monotone and strongly monotone with respect to commodity money, and concave, for $t = 1, 2, \ldots$.[3]

Before to state the next assumptions on utility functions, we need some further definitions introduced by Aumann (1975). Let E be a subset of an Euclidean space. A function $f^i : E \to \mathbb{R}$ is positive on the set E if there is a positive constant c such that $f^i(x) > c$, for each $x \in E$. A function $f^i : E \to \mathbb{R}$ is bounded on the set E if there is a positive constant d such that $f^i(x) < d$, for each $x \in E$. Consider now a sequence of functions $\{f^i(\cdot)\}_i$ such that $f^i : E \to \mathbb{R}$, for each i. A sequence of functions $\{f^i(\cdot)\}_i$ is uniformly positive on the set E if there is a positive constant c such that $f^i(x) > c$, for each $x \in E$, for each i. A sequence of functions $\{f^i(\cdot)\}_i$ is uniformly bounded on the set E if there is a positive constant d such that $f^i(x) < d$, for each $x \in E$, for each i.

Assumption 5 The utility function of a type t trader satisfies the following conditions

(i) $u^t(x) = v^t(x_0, x_t) + z^t(x_1, \ldots, x_{t-1}, x_{t+1}, \ldots)$, for $t = 1, 2, \ldots$;

(ii) the marginal rate of substitution $\frac{\partial v^t}{\partial x_t}(x_0, w_t^t) / \frac{\partial v^t}{\partial x_0}(x_0, w_t^t) < 1$, for each $x_0 \in [0, w_0^t]$, for $t = 1, 2, \ldots$;

[2] All relevant mathematical definitions and results can be found in the mathematical appendix.

[3] Differentiability should be implicitly understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see Kreps (2012), p. 58). Strongly monotone with respect to commodity money means that if there are two commodity bundles x and y such that $x_j \geq y_j$, for each $j \in J \setminus \{0\}$, and $x_0 > y_0$, then $u^t(x) > u^t(y)$.

(iii) the sequence of marginal rates of substitution $\{\frac{\partial v^t}{\partial x_t}(x_0, x_t)/\frac{\partial v^t}{\partial x_0}(x_0, x_t)\}_t$ is uniformly positive and uniformly bounded on the set $[0, 1 + \sup_j w_j] \times [0, 1 + \sup_j w_j]$;

Assumption 6 There exists two types of trader r and t having the following utility functions

$$u^r(x) = v^r(x_0, x_r) + \sum_{j \neq 0,r} \alpha^j z_j^r(x_j),$$

$$u^t(x) = v^t(x_0, x_t) + \sum_{j \neq 0,t} \beta^j z_j^t(x_j),$$

with $\alpha, \beta \in (0, 1)$. The functions $v^r(\cdot)$ and $v^t(\cdot)$ are such that $\frac{\partial v^r}{\partial x_0}(x_0, x_r)$ and $\frac{\partial v^t}{\partial x_0}(x_0, x_t)$ are bounded on the set $[0, 1 + \sup_j w_j] \times [0, 1 + \sup_j w_j]$. The functions $z_j^r(\cdot)$ and $z_j^t(\cdot)$ are such that $\lim_{x_j \to 0} \frac{\partial z_j^r}{\partial x_j}(x_j) = \infty$, for each $j \in J \setminus \{0, r\}$, and $\lim_{x_j \to 0} \frac{\partial z_j^t}{\partial x_j}(x_j) = \infty$, for each $j \in J \setminus \{0, t\}$.

Assumption 1 imposes conditions on the consumption set which are standard in the literature on infinite economies (see Bewley (1972) and Wilson (1981)). Assumptions 2 and 3 are restrictions on initial endowments. Assumption 2 implies that the vector of aggregate initial endowment has an upper bound. Assumption 3 formalises the notion of multilateral oligopoly and implies that the vector of aggregate initial endowment has also a positive lower bound. Assumption 4 specifies classical restrictions on traders' utility functions. Assumption 5 specifies stronger restrictions on traders' utility functions. In order to make the restrictions on marginal rates of substitution clearer and more transparent, for each trader type t the set of commodities is assumed to be partitioned in two subsets: one subset consisting of commodity money and the commodity owned by the trader and another subset consisting of all other commodities. Assumption 5(i) requires that the utility function is additively separable across the two subsets in the partition. Assumption 5(ii) implies that, in a subset of the consumption set, the marginal utility of commodity money is strictly greater that the marginal utility of the other commodity held by the trader. Assumption 5(iii) applies the notions of uniformly positive and uniformly bounded sequence of functions, introduced by Aumann (1975), on the marginal rate of substitution between the commodity in the trader's initial endowment and commodity money. A similar restriction on marginal rates of substitution was imposed, for the first time, by Khan and Vohra (1988) (see also Anderson and Zame (1998), Donnini and Graziano (2009)). Note that, for each type of trader t, the set $[0, 1+\sup_j w_j] \times [0, 1+\sup_j w_j]$ contains all the quantities of commodities x_0^t and x_t^t that are feasible at a type-symmetric allocation, where all traders of the same type have the same commodity bundle.[4] These stronger assumptions are essential to prove that price vectors are uniformly bounded away from zero and from above at any Cournot-Nash equilibria, a fundamental result in the literature on strategic market games. It is worth noting that all the stronger assumptions are made on the function $v^t(\cdot)$ while the function $z^t(\cdot)$ just needs to satisfy the classical restrictions on utility functions. Finally, Assumption 6 is required to show that all commodities are exchanged at the Cournot-Nash equilibrium. Dubey and Shubik (1978) do not make such an assumption because their existence theorem allows for Cournot-Nash equilibria at which some commodities are not exchanged: a further discussion follows in the next section. Finally, we point out that the requirements that the marginal utility of commodity money is bounded for the types of trader satisfying Assumption 6 is

[4]We add 1 to $\sup_j w_j$ because in the perturbed strategic market game (defined in Section 3) the total amount of a commodity may be larger that $\sup_j w_j$.

consistent with the restrictions imposed in Assumptions 5(ii) and 5(iii) (a point clarified in Example 2 below).

We now introduce the strategic market game ${}_k\Gamma$ associated with the exchange economy, with k the number of traders of each type.[5] In this game, each trader has two types of actions: the offer of the commodity in the initial endowment and the bids of commodity money on all other commodities. So, the strategy set of a trader i of type t is

$$S^i = \Big\{ s^i = (q^i_t, b^i_1, \ldots, b^i_{t-1}, b^i_{t+1}, \ldots) : 0 \leq q^i_t \leq w^i_t,\ b^i_j \geq 0,\ \text{for each } j \in J \setminus \{0, t\}, \text{ and } \sum_{j \neq 0,t} b^i_j \leq w^i_0 \Big\},$$

where q^i_t is the offer of commodity t that trader i puts up in exchange for commodity money and b^i_j is the bid of commodity money that he makes on commodity j. Without loss of generality, we make the following technical assumption on the strategy set.

Assumption 7 The set S^i belongs to ℓ^+_∞ endowed with the product topology, for each $i \in I$.

This assumption implies that S^i lies in a normed space and therefore in a Hausdorff space. Let $S = \prod_{i \in I} S^i$ and $S^{-r} = \prod_{i \in I \setminus \{r\}} S^i$. Let s and s^{-i} be elements of S and S^{-i} respectively.

In the game, there is a trading post for each commodity where its price is determined and the commodity is exchanged for commodity money. For each $s \in S$, the price vector $p(s)$ is such that

$$p_j(s) = \begin{cases} \frac{\bar{b}_j}{\bar{q}_j} & \text{if } \bar{q}_j \neq 0 \\ 0 & \text{if } \bar{q}_j = 0 \end{cases},$$

for each $j \in J \setminus \{0\}$, with $\bar{q}_j = \sum_{i \in T_j} q^i_j$ and $\bar{b}_j = \sum_{i \in I \setminus T_j} b^i_j$. By Assumption 2, the sums $\bar{q}_j$ and $\bar{b}_j$ are uniformly bounded from above. For each $s \in S$, the final holding $x^i(s)$ of a trader i of type t is such that

$$x^i_0(s) = w^i_0 - \sum_{j \neq 0,t} b^i_j + q^i_t p_t(s), \tag{1}$$

$$x^i_t(s) = w^i_t - q^i_t, \tag{2}$$

$$x^i_j(s) = \begin{cases} \frac{b^i_j}{p_j(s)} & \text{if } p_j(s) \neq 0 \\ 0 & \text{if } p_j(s) = 0 \end{cases}, \tag{3}$$

for each $j \in J \setminus \{0, t\}$.

The payoff function of a trader i, $\pi^i : S \to \mathbb{R}$, is such that $\pi^i(s) = u^i(x^i(s))$.

We now introduce the definitions of a best response correspondence and a Cournot-Nash equilibrium.

Definition 1 The best response correspondence of a trader i is a correspondence $\phi^i : S^{-i} \to S^i$ such that

$$\phi^i(s^{-i}) \in \underset{s^i \in S^i}{\arg\max} : \pi^i(s^i, s^{-i}),$$

for each $s^{-i} \in S^{-i}$.

[5]Our game extends the model of Shubik (1973) to an infinite dimensional commodity space. The game defined by Dubey and Shubik (1978) differs from ours and Shubik (1973) as in the former traders may bid on all trading posts (including the trading post where the commodity owned by the trader is offered).

Definition 2 An $\hat{s} \in S$ is a Cournot-Nash equilibrium of ${}_k\Gamma$ if $\hat{s}^i \in \phi^i(\hat{s}^{-i})$, for each $i \in I$.

Finally, we call *type-symmetric Cournot-Nash equilibrium* a Cournot-Nash equilibrium in which all traders of the same type play the same strategy and *active Cournot-Nash equilibrium* a Cournot-Nash equilibrium in which $\bar{q}_j > 0$ and $\bar{b}_j > 0$, for each $j \in J \setminus \{0\}$.

3. EXISTENCE OF AN ACTIVE COURNOT-NASH EQUILIBRIUM

In this section we provide an example which clarifies the role of Assumption 6 (Example 1), we state and prove the theorem of existence of an active Cournot-Nash equilibrium, and we show an exchange economy which satisfies Assumptions 1–7 (Example 2).

The overall structure of our existence proof is similar to the one developed in the previous literature on strategic market games (see Dubey and Shubik (1978), Amir et al. (1990), Sahi and Yao (1989)). This consists in proving the existence of a Cournot-Nash equilibrium in a perturbed strategic market game and then in showing that the Cournot-Nash equilibrium of the game ${}_k\Gamma$ is the limit of the sequence of perturbed Cournot-Nash equilibria. This approach relies on the fact that payoff functions are continuous at the limit, so that it is crucial to prove that prices are uniformly bounded away from zero and from above along the sequence of perturbed equilibria (Lemma 3). Dubey and Shubik (1978) showed this result by applying the Uniform Monotonicity Lemma but in our framework we cannot follow that strategy even if such lemma can be proved in our infinite dimensional commodity space. The problem is that prices' lower bounds obtained in their paper depend on the number of commodities and converge to zero in economies with our commodity space.[6] For this reason we develop a new proof, which is inspired by the one adopted in Amir et al. (1990), and it is based on the fact that all commodities are put up in exchange for commodity money at any perturbed Cournot-Nash equilibrium (Lemma 2). This result does not have any analogue in the literature and it is based on the Kuhn-Tucker Theorem for infinite dimensional spaces and on Assumption 5(ii). With this new technique we show that the lower and upper bounds of a price p_j depend on the type j trader's marginal rate of substitution between commodity j and commodity money. This is where the more restrictive Assumption 5(iii) is required. Finally, Assumption 6 is needed to show that the Cournot-Nash equilibrium of ${}_k\Gamma$ is active. It is important to stress that our Theorem 1 is a strengthened version of the existence result proposed by Dubey and Shubik (1978) because under their set of assumptions it is possible to define exchange economies where the unique Cournot-Nash equilibrium has no trade (see Cordella and Gabszewicz (1998)). This is the reason why Dubey and Shubik (1978) requires neither a condition similar to Assumption 6 nor differentiable utility functions. Differently, Bloch and Ferrer (2001) proved the existence of an active Cournot-Nash equilibrium for the bilateral oligopoly model by making assumptions on the limiting behaviours of marginal utilities comparable to the assumption made here.

The following example shows that an exchange economy satisfying Assumptions 1-5 and 7, but not Assumption 6, could have a unique Cournot-Nash equilibrium with no trade.

Example 1 Consider an exchange economy with two traders of each type in which traders of types 1, 2, $r \geq 3$ odd, and $t \geq 4$ even have the following utility functions and initial

[6] See the definition of the lower bound A at page 9 in Dubey and Shubik (1978).

endowments

$$
\begin{aligned}
u^1(x) &= \frac{2}{3}x_0 + \frac{1}{2}x_1 + x_2 & \qquad w^1 &= (2, 2, 0, \dots), \\
u^2(x) &= \frac{2}{3}x_0 + x_1 + \frac{1}{2}x_2 & \qquad w^2 &= (2, 0, 2, 0, \dots), \\
u^r(x) &= \frac{2}{3}x_0 + \frac{1}{2}x_r + x_{r+1} & \qquad w^r &= (2^{-r}, 0, \dots, 0, 2, 0, \dots), \\
u^t(x) &= \frac{2}{3}x_0 + x_{t-1} + \frac{1}{2}x_t & \qquad w^t &= (2^{-t}, 0, \dots, 0, 2, 0, \dots).
\end{aligned}
$$

This exchange economy satisfies Assumptions 1–5 and the strategy sets of the strategic market game satisfy Assumption 7. Moreover, the unique Cournot-Nash equilibrium of the game ${}_2\Gamma$ associated to the exchange economy has no trade.

PROOF: Assumptions 1 and 7 are restrictions on the consumption set and the strategy sets which are satisfied by construction. Since $w_0 = \frac{17}{4}$ and $w_j = 2$, for each $j \in J \setminus \{0\}$, w has a upper bound and then $w \in \ell_\infty^+$. Hence, Assumption 2 is satisfied. Moreover, since $w_0^t > 0$, $w_t^t = 2$, and $w_j^t = 0$, for each $j \in J \setminus \{0, t\}$, for $t = 1, 2, \dots$, Assumption 3 is satisfied. Since each utility function is linear and it depends only on the quantities of three commodities, Assumption 4 is satisfied. Moreover, since $v^r(x_0, x_r) = \frac{2}{3}x_0 + \frac{1}{2}x_r$, $z^r(x_1, \dots, x_{r-1}, x_{r+1} \dots) = x_{r+1}$, for each r odd, and $v^t(x_0, x_t) = \frac{2}{3}x_0 + \frac{1}{2}x_t$, $z^t(x_1, \dots, x_{t-1}, x_{t+1} \dots) = x_{t-1}$, for each t even, Assumption 5(i) is satisfied. Finally, since $\frac{\partial v^t}{\partial x_t}(x_0, x_t) \big/ \frac{\partial v^t}{\partial x_0}(x_0, x_t) = \frac{3}{4}$, for each $x \in X$, for $t = 1, 2, \dots$, Assumptions 5(ii) and 5(iii) are satisfied.

We now show that there exists a unique Cournot-Nash equilibrium at which there is no trade. First, it is straightforward to verify that the Cournot-Nash equilibrium must be type-symmetric. For this reason, in the rest of the proof superscripts refer to trader types. It is important to keep in mind that there are two traders for each type. We now proceed by contradiction. Consider, without loss of generality, the trading post for commodity 1. Suppose that $\bar{\hat{q}}_1 = 2\hat{q}_1^1 > 0$ and $\bar{\hat{b}}_1 = 2\hat{b}_1^2 > 0$. Then, by the necessary conditions of the Generalised Kuhn-Tucker Theorem, we have that

$$
\frac{\partial \pi^1}{\partial q_1^1}(\hat{s}) = \frac{2}{3}\frac{\bar{\hat{b}}_1\bar{\hat{q}}_1 - \bar{\hat{b}}_1\hat{q}_1^1}{(\bar{\hat{q}}_1)^2} - \frac{1}{2} \geq 0 \quad \text{and} \quad \frac{\partial \pi^2}{\partial b_1^2}(\hat{s}) = -\frac{2}{3} + \frac{\bar{\hat{b}}_1\bar{\hat{q}}_1 - \hat{b}_1^2\bar{\hat{q}}_1}{(\bar{\hat{b}}_1)^2} \geq 0.
$$

Since the Cournot-Nash equilibrium is type-symmetric, the inequalities above become

$$
\frac{\hat{b}_1^2}{\hat{q}_1^1} \geq \frac{3}{2} \quad \text{and} \quad \frac{\hat{b}_1^2}{\hat{q}_1^1} \leq \frac{3}{4},
$$

a contradiction. Hence, $\bar{\hat{q}}_1 = 0$ and $\bar{\hat{b}}_1 = 0$. Since this contradiction arises for each commodity, we can conclude that at the unique Cournot-Nash equilibrium there is no trade. *Q.E.D.*

This example clarifies that the existence of an active Cournot-Nash equilibrium relies crucially on the fact that for each commodity j there exists a type of trader t such that $\lim_{x_j \to 0} \frac{\partial u^t}{\partial x_j}(x_j) = \infty$. Therefore, Assumption 6 can be replaced by another assumption as long as it requires this condition to be satisfied. We now state and prove the existence theorem.

THEOREM 1 Under Assumptions 1–7, there exists an active type-symmetric Cournot-Nash equilibrium for ${}_k\Gamma$.

Following Dubey and Shubik (1978), in order to prove the existence of a Cournot-Nash equilibrium, we introduce the perturbed strategic market game ${}_k\Gamma^\epsilon$, the set $Y^i(s^{-i}, \epsilon)$, and the function $x_0^i(x_1^i, x_2^i, \dots)$.[7] The perturbed strategic market game ${}_k\Gamma^\epsilon$ is a game defined as ${}_k\Gamma$ with the only exception that the price vector $p(s)$ becomes

$$p_j^\epsilon(s) = \frac{\bar{b}_j + \epsilon}{\bar{q}_j + \epsilon},$$

for each $j \in J \setminus \{0\}$, with $\epsilon \in (0, 1]$. The interpretation is that an outside agency places a fixed bid of ϵ and a fixed offer of ϵ in each trading post. This does not change the traders' strategy sets, but does affect the prices, the final holdings, and the payoffs. Consider, without loss of generality, a trader i of type t and let

$$Y^i(s^{-i}, \epsilon) = \Big\{(x_1^i, x_2^i, \dots) \in \ell_\infty^+ : x_t^i = w_t^i - q_t^i,\ x_j^i = b_j^i \frac{\bar{q}_j + \epsilon}{\bar{b}_j^i + b_j^i + \epsilon},\ \text{for each } j \in J \setminus \{0, t\},$$
$$\text{for each } s^i \in S^i\Big\},$$

for each $s^{-i} \in S^{-i}$ and $\epsilon \in (0, 1]$, and let

$$x_0^i(x_1^i, x_2^i, \dots) = w_0^i + \sum_{j \neq 0, t} \frac{(\bar{b}_j^i + \epsilon) x_j^i}{x_j^i - \bar{q}_j - \epsilon} + \frac{(\bar{b}_t + \epsilon)(w_t^i - x_t^i)}{\bar{q}_t^i + \epsilon + w_t^i - x_t^i}, \tag{4}$$

with $\bar{q}_t^i = \bar{q}_t - q_t^i$ and $\bar{b}_j^i = \bar{b}_j - b_j^i$. The function $x_0^i(x_1^i, x_2^i, \dots)$ can be easily obtained by the function $x_0^i(s)$ in (1) by relabelling the variables. The set $Y^i(s^{-i}, \epsilon)$ contains all the commodity bundles $(x_1^i, x_2^i, \dots)$ that are feasible final holdings for the trader i in the perturbed strategic market game, for any given $s^{-i} \in S^{-i}$ and $\epsilon \in (0, 1]$. It is worth noting that the quantity of commodity money is not included in the commodity bundles belonging to $Y^i(s^{-i}, \epsilon)$. Before we start with the proof of existence, we prove two preliminary results on $Y^i(s^{-i}, \epsilon)$ and $x_0^i(x_1^i, x_2^i, \dots)$ which are used in Lemma 1.

Proposition 1 The set $Y^i(s^{-i}, \epsilon)$ is convex, for each $i \in I$.

Proof: Consider, without loss of generality, a trader i of type t and fix the strategies s^{-i} for all other traders. Take two commodity bundles $x'^i, x''^i \in Y^i(s^{-i}, \epsilon)$ and consider $x^{*i} = \lambda x'^i + (1 - \lambda) x''^i$.[8] We want to show that $x^{*i} \in Y^i(s^{-i}, \epsilon)$. Then, there must exist a strategy $s^{*i} \in S^i$ such that $x^i(s^{*i}, s^{-i}) = x^{*i}$. Let $x'^i = x^i(s'^i, s^{-i})$ and $x''^i = x^i(s''^i, s^{-i})$. Consider first the commodity t. Since the function in (2) is linear in q_t^i, we have that $x_t^{*i} = x_t^i(\lambda q_t'^i + (1 - \lambda) q_t''^i, s^{-i}) = x_t^i(q_t^{*i}, s^{-i})$.[9] Consider now a commodity $j \neq t$. Since the function in (3) is concave in b_j^i, we obtain that

$$x_j^{*i} = \lambda x_j'^i + (1 - \lambda) x_j''^i = \lambda x_j^i(b_j'^i, s^{-i}) + (1 - \lambda) x_j^i(b_j''^i, s^{-i}) \leq x_j^i(\lambda b_j'^i + (1 - \lambda) b_j''^i, s^{-i}).$$

By the Intermediate Value Theorem and since $x_j^i(\lambda b_j'^i + (1 - \lambda) b_j''^i, s^{-i}) = 0$ by setting $b_j'^i = 0$ and $b_j''^i = 0$, we may reduce $b_j'^i$ and $b_j''^i$ appropriately to get x_j^{*i}. Then, there exists b_j^{*i} such that $x_j^{*i} = x_j^i(b_j^{*i}, s^{-i})$, for each $j \in J \setminus \{0, t\}$. Hence, there exists a $s^{*i} \in S^i$ such that $x^{*i} = x^i(s^{*i}, s^{-i})$ and then $x^{*i} \in Y^i(s^{-i}, \epsilon)$. *Q.E.D.*

[7] Dubey and Shubik (1978) denotes the set $Y^i(s^{-i}, \epsilon)$ with $D^i(Q, B, \epsilon)$. In order to save in notation, with some abuse, we denote by $x_0^i(\cdot)$ both the function $x_0^i(s)$ and the function $x_0^i(x_1^i, x_2^i, \dots)$.

[8] It is important to keep in mind that these commodity bundles do not include the quantity of commodity money.

[9] To clarify the exposition, in this proof we write $x_t^i(q_t^i, s^{-i})$ and $x_j^i(b_j^i, s^{-i})$ instead of $x_t^i(s^i, s^{-i})$ and $x_j^i(s^i, s^{-i})$, for each $j \in J \setminus \{0, t\}$.

Proposition 2 The function $x_0^i(x_1^i, x_2^i, \dots)$ is strictly concave on the set $Y^i(s^{-i}, \epsilon)$, for each $i \in I$.

Proof: Consider, without loss of generality, a trader i of type t and fix the strategies s^{-i} for all other traders. It is straightforward to verify that each term of the summation in (4) is a strictly concave function, i.e., $\frac{(\bar{b}_j^i+\epsilon)x_j^i}{x_j^i-\bar{q}_j-\epsilon}$ is strictly concave in x_j^i, for each $x_j^i \in [0, \bar{q}_j]$, for each $j \in J\setminus\{0,t\}$, and $\frac{(\bar{b}_t+\epsilon)(w_t^i-x_t^i)}{\bar{q}_t^i+\epsilon+w_t^i-x_t^i}$ is strictly concave in x_t^i, for each $x_t^i \in [0, w_t^i]$. Since each term of the summation is a strictly concave function, we obtain the following inequality

$$x_0^i(\lambda x_1'^i + (1-\lambda)x_1''^i, \lambda x_2'^i + (1-\lambda)x_2''^i, \dots) = w_0^i + \sum_{j\neq 0,t} \frac{(\bar{b}_j^i+\epsilon)(\lambda x_j'^i + (1-\lambda)x_j''^i)}{(\lambda x_j'^i + (1-\lambda)x_j''^i) - \bar{q}_j - \epsilon} +$$

$$\frac{(\bar{b}_t+\epsilon)(w_t^i - (\lambda x_t'^i + (1-\lambda)x_t''^i))}{\bar{q}_t^i+\epsilon+w_t^i - (\lambda x_t'^i + (1-\lambda)x_t''^i)} > w_0^i + \sum_{j\neq 0,t}\left(\lambda\frac{(\bar{b}_j^i+\epsilon)x_j'^i}{x_j'^i-\bar{q}_j-\epsilon} + (1-\lambda)\frac{(\bar{b}_j^i+\epsilon)x_j''^i}{x_j''^i-\bar{q}_j-\epsilon}\right) +$$

$$\lambda\frac{(\bar{b}_t+\epsilon)(w_t^i-x_t'^i)}{\bar{q}_t^i+\epsilon+w_t^i-x_t'^i} + (1-\lambda)\frac{(\bar{b}_t+\epsilon)(w_t^i-x_t''^i)}{\bar{q}_t^i+\epsilon+w_t^i-x_t''^i} = \lambda x_0^i(x_1'^i, x_2'^i, \dots) + (1-\lambda)x_0^i(x_1''^i, x_2''^i, \dots).$$

But then, $x_0^i(x_1^i, x_2^i, \dots)$ is strictly concave on the set $Y^i(s^{-i}, \epsilon)$. *Q.E.D.*

In the next lemma we prove that there exists a type-symmetric Cournot-Nash equilibrium in the perturbed strategic market game by using a fixed point theorem for infinite dimensional spaces.

Lemma 1 There exists a type-symmetric Cournot-Nash equilibrium of ${}_k\Gamma^\epsilon$, for each $\epsilon \in (0, 1]$.

Proof: Consider, without loss of generality, a trader i and fix the strategies s^{-i} for all other traders. In the perturbed game the payoff function $\pi^i(\cdot)$ is continuous because it is a composition of continuous functions (see Theorem 17.23, p. 566 in Aliprantis and Border (2006), AB hereafter). The definition of the strategy set S^i and Assumptions 2, 3, and 7 imply that S^i is a non-empty and compact set. The compactness follows straightforwardly by the Tychonoff Theorem (see Theorem 2.61, p. 52 in AB). Then, there exists a strategy in S^i that maximises the payoff function $\pi^i(\cdot)$ by the Weierstrass Theorem (see Corollary 2.35, p. 40 in AB). Hence, the best response correspondence $\phi^i : S^{-i} \to S^i$ is non-empty. Moreover, since S^i belongs to a Hausdorff space by Assumption 7, the correspondence $\phi^i(\cdot)$ is upper hemicontinuous by the Berge Maximum Theorem (see Theorem 17.31, p. 570 in AB). We now refine this result by showing that $\phi^i(\cdot)$ is actually a continuous function. Suppose that there are two strategies s'^i and s''^i such that the final holdings $x^i(s'^i, s^{-i}) = x'^i$ and $x^i(s''^i, s^{-i}) = x''^i$ maximise the payoff function. Consider the commodity bundle $x^{*i} = \frac{1}{2}x'^i + \frac{1}{2}x''^i$. Since the utility function is concave, we have that $u^i(x^{*i}) \geq \frac{1}{2}u^i(x'^i) + \frac{1}{2}u^i(x''^i) = u^i(x'^i)$. Since $Y^i(s^{-i}, \epsilon)$ is convex, by Proposition 1, and $x_0^i(x_1^i, x_2^i, \dots)$ is strictly concave, by Proposition 2, there exists a $\gamma > 0$ such that the commodity bundle $x^{*i} + \gamma e_0$ is a feasible final holding. That is, there exists a strategy $s^{*i} \in S^i$ such that $x^{*i} + \gamma e_0 = x^i(s^{*i}, s^{-i})$.[10] Then, as the utility function is strongly monotone with respect to x_0^i by Assumption 4, we obtain that $u^i(x^{*i} + \gamma e_0) > u^i(x'^i)$, a contradiction. But then, there is only one strategy that maximises the payoff function and $\phi^i(\cdot)$ is a single-valued best response correspondence. Hence, $\phi^i(\cdot)$ is a continuous function (see Lemma 17.6, p. 559 in AB), for each $i \in I$. As we are looking for a fixed point in the strategy space S, let us consider S as the domain

[10] e_j is an infinite vector in ℓ_∞ whose jth component is 1 and all others are 0.

of the best response function, i.e., $\phi^i : S \to S^i$. We then define a function $\Phi : S \to S$ such that $\Phi(s) = \prod_{i \in I} \phi^i(s)$. The function $\Phi(\cdot)$ is continuous since it is a product of continuous functions (see Theorem 17.28, p. 568 in AB). Moreover, the strategy space S belongs to a Hausdorff space and it is a non-empty, compact, and convex set as it is a product of non-empty, compact, and convex sets. As above, compactness follows by the Tychonoff Theorem. Then, the Brouwer-Schauder-Tychonoff Theorem (see Corollary 17.56, p. 583 in AB) implies that there exists a fixed point $\hat{s}$ of $\Phi(\cdot)$, which is a Cournot-Nash equilibrium of the perturbed game ${}_k\Gamma^\epsilon$. We refine this result by showing that there exists a type-symmetric Cournot-Nash equilibrium. Let ${}_kS$ be the set of type-symmetric strategy profiles in S. It is immediate to verify that ${}_kS$ is a non-empty, compact, and convex set in a Hausdorff space. If $\Phi(\cdot)$ is defined over the domain ${}_kS$, $\Phi : {}_kS \to S$, $\Phi(\cdot)$ is still a continuous function. Furthermore, in a type-symmetric situation traders of the same type face the same optimisation problem and then $\Phi(s) \in {}_kS$, for each $s \in {}_kS$. Then, there exists a fixed point $\hat{s}$ of $\Phi : {}_kS \to S$, by the Brouwer-Schauder-Tychonoff Theorem, and $\hat{s} \in {}_kS$. Hence, $\hat{s}$ is a type-symmetric Cournot-Nash equilibrium of the perturbed game ${}_k\Gamma^\epsilon$. *Q.E.D.*

Lemma 2 shows that all commodities are put up in exchange for commodity money at a type-symmetric Cournot-Nash equilibrium of the perturbed game. The proof is based on the Generalised Kuhn-Tucker Theorem and it requires Assumption 5(ii). Previous contributions on strategic market games in finite economies do not need this result to prove the existence.

Lemma 2 At any type-symmetric Cournot-Nash equilibrium $\hat{s}$ of the perturbed game, $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$.

Proof: Let $\hat{s}$ be a type-symmetric Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i of type t. Since $\hat{s}^i$ belongs to a Cournot-Nash equilibrium, it solves the following maximisation problem

$$\begin{aligned}
\max_{s^i} \quad & \pi^i(s^i, \hat{s}^{-i}), & \\
\text{subject to} \quad & q_t^i \leq w_t^i, & (i) \\
& \sum_{j \neq 0,t} b_j^i \leq w_0^i, & (ii) \\
& -q_t^i \leq 0, & (iii) \\
& -b_j^i \leq 0, \text{ for each } j \in J \setminus \{0,t\}. & (iv)
\end{aligned} \tag{5}$$

The constraints can be written as a function $g : \ell_\infty \to Z$, with $Z \subset \ell_\infty$. It is straightforward to verify that Z contains a closed positive cone with a non-empty interior and $g(\cdot)$ is Fréchet differentiable. We now show that there exists an $h \in \ell_\infty$ such that

$$g(\hat{s}^i) + g'(\hat{s}^i)h < 0,$$

with $g'(\cdot)$ the Fréchet derivative of $g(\cdot)$. That is, we prove that $\hat{s}^i$ is a regular point of the constrained set.[11] Given the constraints in (5), the inequality above can be written as the

[11]The notion of regular point, rigorously defined in the mathematical appendix, is a type of constraint qualification (see Exercise 3 of Chapter 9 in Luenberger (1969)) and it does not correspond to the classical Kuhn-Tucker constraint qualification (see Exercise 7 of Chapter 9 in Luenberger (1969)).

following vector inequality

$$\begin{bmatrix} \hat{q}^i_t - w^i_t \\ \sum_{j\neq 0,t} \hat{b}^i_j - w^i_0 \\ -\hat{q}^i_t \\ -\hat{b}^i_1 \\ -\hat{b}^i_2 \\ \dots \end{bmatrix} + \begin{bmatrix} h_t \\ \sum_{j\neq 0,t} h_j \\ -h_t \\ -h_1 \\ -h_2 \\ \dots \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{bmatrix}. \tag{6}$$

First, suppose that the constraints (i) and (ii) are not binding. Consider a vector h with h_t positive and sufficiently small, h_j positive and sufficiently small, for each j such that $\hat{b}^i_j = 0$, and $h_j = 0$, for each j such that $\hat{b}^i_j > 0$. Then, the vector inequality (6) is satisfied and $\hat{s}^i$ is a regular point. Now, suppose that the constraints (i) and (ii) are binding. Consider a vector h with h_t negative and sufficiently small, h_j negative and sufficiently small, for each j such that $\hat{b}^i_j > 0$, and h_j positive and sufficiently small, for each j such that $\hat{b}^i_j = 0$. Then, the vector inequality (6) is satisfied and $\hat{s}^i$ is a regular point. If either constraint (i) or (ii) is binding, the previous argument leads, *mutatis mutandis*, to the same result. Hence, $\hat{s}^i$ is a regular point of the constrained set. Finally, since we consider a perturbed strategic market game and the utility function is Fréchet differentiable by Assumption 4, the payoff function $\pi^i(\cdot)$ is Fréchet differentiable as it is a composition of Fréchet differentiable functions. We have then proved that all the hypothesis of the Generalised Kuhn-Tucker Theorem are satisfied (see the mathematical appendix). Therefore, there exist non-negative multipliers $\hat{\lambda}^i_1$ and $\hat{\mu}^i_t$ such that

$$\begin{aligned} &\frac{\partial \pi^i}{\partial q^i_t}(\hat{s}^i, \hat{s}^{-i}) - \hat{\lambda}^i_1 + \hat{\mu}^i_t = 0, \\ &\hat{\lambda}^i_1(\hat{q}^i_t - w^i_t) = 0, \\ &\hat{\mu}^i_t \hat{q}^i_t = 0. \end{aligned} \tag{7}$$

By the payoff function definition and Assumption 5(i), equation (7) can be written as

$$\frac{\partial v^i}{\partial x_0}(x^i_0(\hat{s}), x^i_t(\hat{s}))\frac{\bar{\hat{b}}_t + \epsilon}{\bar{\hat{q}}_t + \epsilon}\left(1 - \frac{\hat{q}^i_t}{\bar{\hat{q}}_t + \epsilon}\right) - \frac{\partial v^i}{\partial x_t}(x^i_0(\hat{s}), x^i_t(\hat{s})) - \hat{\lambda}^i_1 + \hat{\mu}^i_t = 0. \tag{8}$$

We now proceed by contradiction and we suppose that $\bar{\hat{q}}_t = 0$. Then, $\hat{q}^i_t = 0$ which implies that $x^i_t(\hat{s}) = w^i_t$, by equation (2), and $\hat{\lambda}^i_1 = 0$, by the complementary slackness conditions as the constraint (i) is not binding. But then, the previous equation becomes

$$\frac{\partial v^i}{\partial x_0}(x^i_0(\hat{s}), w^i_t)\frac{\bar{\hat{b}}_t + \epsilon}{\epsilon} - \frac{\partial v^i}{\partial x_t}(x^i_0(\hat{s}), w^i_t) + \hat{\mu}^i_t = 0,$$

with $x^i_0(\hat{s}) \in [0, w^i_0]$, by equation (1). Since $\frac{\bar{\hat{b}}_t+\epsilon}{\epsilon} \geq 1$ and $\frac{\partial v^i}{\partial x_t}(x_0, w^i_t) < \frac{\partial v^i}{\partial x_0}(x_0, w^i_t)$ for each $x_0 \in [0, w^i_0]$, by Assumption 5(ii), the left hand side of the equation is greater than zero, a contradiction. Hence, $\bar{\hat{q}}_t > 0$. We can then conclude that $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$. *Q.E.D.*

We now show that price vectors at type-symmetric Cournot-Nash equilibria have a uniform positive lower bound and a uniform upper bound independent of ϵ and k. Since the number of commodities is infinite, we cannot apply the same approach adopted by Dubey and Shubik (1978). To prove the next lemma is essential Assumption 5(iii) and the result obtained in Lemma 2.

Lemma 3 For any type-symmetric Cournot-Nash equilibrium $\hat{s}$ of the perturbed game, there exist two positive constants C and D, independent of ϵ and k, such that

$$C < p_j^\epsilon(\hat{s}) < D,$$

for each $j \in J \setminus \{0\}$.

Proof: For a trader of type t, it is immediate to verify that $(x_0^t(\hat{s}), x_t^t(\hat{s})) \in [0, 1 + \sup_j w_j] \times [0, 1 + \sup_j w_j]$, at any type-symmetric Cournot-Nash equilibrium $\hat{s}$ of the perturbed strategic market game, for $t = 1, 2, \ldots$, i.e., the set above contains all the quantities of the commodities 0 and t which can be obtained at a final holding of a type-symmetric Cournot-Nash equilibrium. Since the sequence $\{\frac{\partial v^t}{\partial x_t}(x_0, x_t) \big/ \frac{\partial v^t}{\partial x_0}(x_0, x_t)\}_t$ is uniformly positive and uniformly bounded on the set $[0, 1 + \sup_j w_j] \times [0, 1 + \sup_j w_j]$ by Assumption 5(iii), there exist two positive constants C' and D', independent of ϵ and k, such that

$$C' < \frac{\partial v^t}{\partial x_t}(x_0^t(\hat{s}), x_t^t(\hat{s})) \bigg/ \frac{\partial v^t}{\partial x_0}(x_0^t(\hat{s}), x_l^t(\hat{s})) < D', \tag{9}$$

for any type-symmetric Cournot-Nash equilibrium $\hat{s}$, for each type of trader $t = 1, 2, \ldots$. Consider now, without loss of generality, a type-symmetric Cournot-Nash equilibrium $\hat{s}$ of ${}_k\Gamma^\epsilon$. We first establish the existence of C. By Lemma 2, there exists a trader i of type l such that $\hat{q}_l^i > 0$. Then, a decrease γ in i's offer of commodity l is feasible, with $0 < \gamma \leq \hat{q}_l^i$, and has the following incremental effects on the final holding of trader i

$$\begin{aligned}
x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) &= (\hat{q}_l^i - \gamma)\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon - \gamma} - \hat{q}_l^i \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}, \\
&= \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}\left((\hat{q}_l^i - \gamma)\frac{\bar{\hat{q}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon - \gamma} - \hat{q}_l^i\right) \geq -p_l^\epsilon(\hat{s})\gamma, \\
x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) &= \gamma, \\
x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) &= 0, \text{ for each } j \in J \setminus \{0, l\}.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\bar{\hat{q}}_l + \epsilon > \bar{\hat{q}}_l + \epsilon - \gamma$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) - p_l^\epsilon(\hat{s})\gamma e_0 + \gamma e_l.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq -\frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}), x_l^i(\hat{s}))p_l^\epsilon(\hat{s})\gamma + \frac{\partial v^i}{\partial x_l}(x_0^i(\hat{s}), x_l^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p_l^\epsilon(\hat{s}) > \frac{\partial v^i}{\partial x_l}(x_0^i(\hat{s}), x_l^i(\hat{s})) \bigg/ \frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}), x_l^i(\hat{s})).$$

By the inequalities in (9), we have that $p_l^\epsilon(\hat{s}) > C'$. We can then choose $C = C'$ and conclude that $p_j^\epsilon(\hat{s}) > C$, for each $j \in J \setminus \{0\}$, for any type-symmetric Cournot-Nash equilibrium $\hat{s}$, with C independent of ϵ and k.

Now, we establish the existence of D. Since there are at least two traders of each type, we consider a trader i of type l such that $\hat{q}_l^i \leq \frac{\bar{\hat{q}}_l}{2}$. We need to consider two cases. First, suppose that $\hat{q}_l^i < w_l^i$. Then, an increase γ in i's offer of commodity l is feasible, with

$0 < \gamma < \min\{w_l^i - \hat{q}_l^i, \epsilon\}$, and has the following incremental effects on the final holding of trader i

$$\begin{aligned} x_0^i(\hat{s}(\gamma)) - x_0^i(\hat{s}) &= (\hat{q}_l^i + \gamma)\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon + \gamma} - \hat{q}_l^i \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}, \\ &= \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon} \frac{\bar{\hat{q}}_l^i + \epsilon}{\bar{\hat{q}}_l^i + \hat{q}_l^i + \epsilon + \gamma}\gamma \geq \frac{1}{3} p_l^\epsilon(\hat{s})\gamma, \\ x_l^i(\hat{s}(\gamma)) - x_l^i(\hat{s}) &= -\gamma, \\ x_j^i(\hat{s}(\gamma)) - x_j^i(\hat{s}) &= 0, \text{ for each } j \in J \setminus \{0, l\}. \end{aligned}$$

The inequality in the preceding array follows from the fact that $\hat{q}_l^i \leq \bar{\hat{q}}_l^i + \epsilon$ and $\gamma \leq \bar{\hat{q}}_l^i + \epsilon$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) + \frac{1}{3} p_l^\epsilon(\hat{s})\gamma e_0 - \gamma e_l.$$

By using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u^i(x^i(\hat{s}(\gamma))) - u^i(x^i(\hat{s})) \geq \frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}), x_l^i(\hat{s}))\frac{1}{3}p_l^\epsilon(\hat{s})\gamma - \frac{\partial v^i}{\partial x_l}(x_0^i(\hat{s}), x_l^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p_l^\epsilon(\hat{s}) < 3\left(\frac{\partial v^i}{\partial x_l}(x_0^i(\hat{s}), x_l^i(\hat{s})) \Big/ \frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}), x_l^i(\hat{s}))\right).$$

By the inequalities in (9), we have that $p_l^\epsilon(\hat{s}) < 3D'$. Now, suppose that $\hat{q}_l^i = w_l^i$. Then,

$$p_l^\epsilon(\hat{s}) \leq \frac{kw_0 + 1}{kw_l^i} < \frac{w_0 + 1}{w_l^i} < \frac{\sup_j w_j + 1}{W} = D''.$$

The denominator of the first fraction is kw_l^i because the Cournot-Nash equilibrium is type-symmetric and the last inequality follows from the fact that $w_0 \leq \sup_j w_j$, by Assumption 2, and $w_l^i \geq W$, by Assumption 3. It is immediate to see that the constant D'' is independent of ϵ and k. We can then choose $D = \max\{3D', D''\}$ and conclude that $p_j^\epsilon(\hat{s}) < D$, for each $j \in J \setminus \{0\}$, for any type-symmetric Cournot-Nash equilibrium $\hat{s}$, with D independent of ϵ and k. *Q.E.D.*

We are now ready to prove Theorem 1. Since we need to consider a sequence of Cournot-Nash equilibria at different perturbed games, in the next proof, we write $\hat{s}^{\epsilon_n}$ to denote a type-symmetric Cournot-Nash equilibrium of the perturbed game ${}_k\Gamma^{\epsilon_n}$ and $\hat{s}$ to denote the limit of the sequence $\{\hat{s}^{\epsilon_n}\}_n$.[12]

PROOF OF THEOREM 1: Consider a sequence of $\{\epsilon_n\}_n$ converging to 0. By Lemma 1, in each perturbed game there exists a type-symmetric Cournot-Nash equilibrium. Then, we can consider a sequence of type-symmetric Cournot-Nash equilibria $\{\hat{s}^{\epsilon_n}\}_n$ associated to the sequence $\{\epsilon_n\}_n$. As proved in Lemma 1, S is compact and, by Lemma 3, $p_j^{\epsilon_n}(\hat{s}^{\epsilon_n}) \in [C, D]$, which is a compact interval, for each $j \in J \setminus \{0\}$. Then, we can pick a subsequence of $\{\hat{s}^{\epsilon_{h_n}}\}_n$ that converge to s such that $s \in S$ and $p_j(s) \in [C, D]$, for each $j \in J \setminus \{0\}$, as the product topology is the topology of coordinate-wise convergence. But then, the strategy profile s is a point of continuity of payoff functions and then it is a type-symmetric Cournot-Nash

[12]To avoid cumbersome notation, in the proofs of Lemmas 1–3 we have not written $\hat{s}^\epsilon$ even if the Cournot-Nash equilibrium belongs to a game ${}_k\Gamma^\epsilon$.

equilibrium of the game ${}_k\Gamma$, i.e., $\hat{s}$. It remains to prove that $\hat{s}$ is an active type-symmetric Cournot-Nash equilibrium. Consider a trader i of type t satisfying Assumption 6. We know that the strategy $\hat{s}^{i,\epsilon_{h_n}}$ solves the maximization problem (5) as it belongs to a Cournot-Nash equilibrium, for each n. Since all the hypothesis of the Generalised Kuhn-Tucker Theorem are satisfied, as proved in Lemma 2, there exist non-negative multipliers $\hat{\lambda}_2^{i,\epsilon_{h_n}}$ and $\hat{\mu}_j^{i,\epsilon_{h_n}}$, for each $j \in J \setminus \{0, t\}$, such that

$$\frac{\partial \pi^i}{\partial b_j^i}(\hat{s}^{i,\epsilon_{h_n}}, \hat{s}^{-i,\epsilon_{h_n}}) - \hat{\lambda}_2^{i,\epsilon_{h_n}} + \hat{\mu}_j^{i,\epsilon_{h_n}} = 0, \text{ for each } j \in J \setminus \{0, t\}, \tag{10}$$

$$\hat{\lambda}_2^{i,\epsilon_{h_n}} \Big(\sum_{j \neq 0,t} \hat{b}_j^{i,\epsilon_{h_n}} - w_0^i \Big) = 0,$$

$$\hat{\mu}_j^{i,\epsilon_{h_n}} \hat{b}_j^{i,\epsilon_{h_n}} = 0, \text{ for each } j \in J \setminus \{0, t\}.$$

By the definition of the payoff function and Assumption 6, equation (10) can be written as

$$\begin{aligned} -\frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}^{\epsilon_{h_n}}), x_t^i(\hat{s}^{\epsilon_{h_n}})) + \beta^j \frac{\partial z_j^i}{\partial x_j}(x_j^i(\hat{s}^{\epsilon_{h_n}})) \frac{1}{p_j^{\epsilon_{h_n}}(\hat{s}^{\epsilon_{h_n}})} \Bigg(1 - \frac{\hat{b}_j^{i,\epsilon_{h_n}}}{\hat{\bar{b}}_j^{\epsilon_{h_n}} + \epsilon_{h_n}} \Bigg) \\ -\hat{\lambda}_2^{i,\epsilon_{h_n}} + \hat{\mu}_j^{i,\epsilon_{h_n}} = 0, \end{aligned} \tag{11}$$

for each $j \in J \setminus \{0, t\}$, for each n. We now proceed by contradiction. We first suppose that the sequence of trader i's sums of bids converges to zero, i.e., $\lim_{n\to\infty} \sum_{j\neq 0,t} \hat{b}_j^{i,\epsilon_{h_n}} = 0$. Then, there exists a natural number N such that $\sum_{j\neq 0,t} \hat{b}_j^{i,\epsilon_{h_n}} < w_0^i$, for each $n \geq N$. Then, $\lambda_1^{i,\epsilon_{h_n}} = 0$, for each $n \geq N$ by the complementary slackness condition. Consider, without loss of generality, a commodity j. We have that $1 - \frac{\hat{b}_j^{i,\epsilon_{h_n}}}{\hat{\bar{b}}_j^{\epsilon_{h_n}} + \epsilon_{h_n}} \geq \frac{1}{2}$, as the Cournot-Nash equilibrium is type-symmetric and $k \geq 2$, and $\frac{1}{p_j^{\epsilon_{h_n}}(\hat{s}^{\epsilon_{h_n}})} > \frac{1}{D}$, by Lemma 3. Then, we can derive the following inequality from the first order condition (11)

$$-\frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}^{\epsilon_{h_n}}), x_t^i(\hat{s}^{\epsilon_{h_n}})) + \beta^j \frac{\partial z_j^i}{\partial x_j}(x_j^i(\hat{s}^{\epsilon_{h_n}})) \frac{1}{D}\frac{1}{2} + \hat{\mu}_j^{i,\epsilon_{h_n}} \leq 0, \tag{12}$$

which is satisfied at any Cournot-Nash equilibrium $\hat{s}^{\epsilon_{h_n}}$ with $n \geq N$. When $n \to \infty$, we have that $\hat{b}_j^{i,\epsilon_{h_n}} \to 0$, as $\sum_{j\neq 0,t} \hat{b}_j^{i,\epsilon_{h_n}} \to 0$, and this implies that $\frac{\partial z_j^i}{\partial x_j}(x_j^i(\hat{s}^{\epsilon_{h_n}})) \to \infty$ as $\lim_{x_j \to 0} \frac{\partial z_j^i}{\partial x_j}(x_j) = \infty$, by Assumption 6. Since, the marginal utility $\frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}^{\epsilon_{h_n}}), x_t^i(\hat{s}^{\epsilon_{h_n}}))$ has an upper bound, as $\frac{\partial v^i}{\partial x_0}(x_0, x_t)$ is bounded on $[0, 1 + \sup_j w_j] \times [0, 1 + \sup_j w_j]$ by Assumption 6, it follows that there exists a natural number N' such that the left hand side of inequality (12) is positive at $\hat{s}^{\epsilon_{h_n}}$, for each $n \geq N'$, a contradiction. Then, we can conclude that the sequence of trader i's sums of bids converges to a positive constant, i.e., $\lim_{n\to\infty} \sum_{j\neq 0,t} \hat{b}_j^{i,\epsilon_{h_n}} > 0$. This result implies that there exists at least one commodity l such that $\hat{b}_l^i > 0$. Since, $1 - \frac{\hat{b}_l^{i,\epsilon_{h_n}}}{\hat{\bar{b}}_l^{\epsilon_{h_n}} + \epsilon_{h_n}} \leq 1$, $\frac{1}{p_l^{\epsilon_{h_n}}(\hat{s}^{\epsilon_{h_n}})} < \frac{1}{C}$, by Lemma 3, and $\frac{\partial v^i}{\partial x_0}(x_0^i(\hat{s}^{\epsilon_{h_n}}), x_t^i(\hat{s}^{\epsilon_{h_n}}))$ has an upper bound, by Assumption 6, we can derive the following inequality from the first order condition (11) with respect to commodity l

$$\hat{\lambda}_2^{i,\epsilon_{h_n}} \leq \beta^l \frac{\partial z_l^i}{\partial x_l}(x_l^i(\hat{s}^{\epsilon_{h_n}})) \frac{1}{C} + \hat{\mu}_l^{i,\epsilon_{h_n}}. \tag{13}$$

Since $\lim_{n\to\infty} \hat{b}_l^{i,\epsilon_{h_n}} = \hat{b}_l^i > 0$, there exists a natural number N'' such that $\hat{b}_l^{i,\epsilon_{h_n}} > 0$, for each $n \geq N''$. This implies that $\frac{\partial z_l^i}{\partial x_l}(x_l^i(\hat{s}^{\epsilon_{h_n}}))$ has a uniform upper bound, for each $n \geq N''$, as $z^i_j(\cdot)$ is continuously Fréchet differentiable by Assumption 4, and that $\mu_l^{i,\epsilon_{h_n}} = 0$, for each $n \geq N''$, by the complementary slackness condition as constraint (iv) is not binding. Then, $\lim_{n\to\infty} \hat{\lambda}_2^{i,\epsilon_{h_n}} < \infty$. We proceed again by contradiction. Suppose that there exists a commodity $j \neq l, t$ such that $\hat{b}^i_j = 0$. By the same steps used above to obtain inequality (12), we can obtain the following inequality for the commodity j such that $\hat{b}^i_j = 0$

$$-\frac{\partial v^i}{\partial x_0}(x^i_0(\hat{s}^{\epsilon_{h_n}}), x^i_t(\hat{s}^{\epsilon_{h_n}})) + \beta^j \frac{\partial z^i_j}{\partial x_j}(x^i_j(\hat{s}^{\epsilon_{h_n}}))\frac{1}{D}\frac{1}{2} - \hat{\lambda}_2^{i,\epsilon_{h_n}} + \hat{\mu}_j^{i,\epsilon_{h_n}} \leq 0, \tag{14}$$

which is satisfied at any type-symmetric Cournot-Nash equilibrium $\hat{s}^{\epsilon_{h_n}}$, for each n. As above, $\frac{\partial v^i}{\partial x_0}(x^i_0(\hat{s}^{\epsilon_{h_n}}), x^i_t(\hat{s}^{\epsilon_{h_n}}))$ has an upper bound and, when $n \to \infty$, we have that $\frac{\partial z^i_j}{\partial x_j}(x^i_j(\hat{s}^{\epsilon_{h_n}})) \to \infty$ as $\hat{b}_j^{i,\epsilon_{h_n}} \to 0$. Then, it follows that $\lim_{n\to\infty} \hat{\lambda}_2^{i,\epsilon_{h_n}} = \infty$ because the left hand side of inequality (14) must be negative. But, by inequality (13), we have that $\lim_{n\to\infty} \hat{\lambda}_2^{i,\epsilon_{h_n}} < \infty$, a contradiction. Hence, we can conclude that $\hat{b}^i_j > 0$, for each $j \in J \setminus \{0, t\}$. This implies that $\bar{\hat{b}}_j > 0$, for each $j \in J \setminus \{0, t\}$. By considering a trader i of type r satisfying Assumption 6, the previous argument leads, *mutatis mutandis*, to the same result for the bid on commodity t, i.e., $\bar{\hat{b}}_t > 0$. Hence, we have that $\bar{\hat{b}}_j > 0$, for each $j \in J \setminus \{0\}$. Moreover, this result implies that also $\bar{\hat{q}}_j > 0$, for each $j \in J \setminus \{0\}$, because $p_j(\hat{s}) \in [C, D]$ and $p_j(\hat{s}) = \frac{\bar{\hat{b}}_j}{\bar{\hat{q}}_j}$, for each $j \in J \setminus \{0\}$. Therefore, $\hat{s}$ is an active type-symmetric Cournot-Nash equilibrium. *Q.E.D.*

We now show an example where Assumptions 1–7 are satisfied.

Example 2 Consider an exchange economy with k traders of each type in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$u^1(x) = \frac{2}{3}x_0 + \frac{1}{2}x_1 + \sum_{j=2}^{\infty} 3^{-j}\ln x_j \qquad w^1 = (2, 2, 0, \dots),$$

$$u^2(x) = \frac{2}{3}x_0 + \frac{1}{2}x_2 + 3^{-1}\ln x_1 + \sum_{j=3}^{\infty} 3^{-j}\ln x_j \qquad w^2 = (2, 0, 2, 0, \dots),$$

$$u^3(x) = \frac{2}{3}x_0 + \frac{1}{2}x_3 + \sum_{j=4}^{\infty} 3^{-j}\ln x_j \qquad w^3 = (2^{-3}, 0, 0, 2, 0, \dots),$$

$$u^t(x) = \frac{2}{3}x_0 + \frac{1}{2}x_t + \sum_{j=t+1}^{\infty} 3^{-j}\ln x_j \qquad w^t = (2^{-t}, 0, \dots, 0, 2, 0, \dots).$$

This exchange economy satisfies Assumptions 1–6 and the strategy sets of the strategic market game satisfy Assumption 7. Furthermore, the active type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$ associated to the exchange economy is

$$(\hat{q}^1_1, \hat{b}^1_2, \hat{b}^1_3, \dots, \hat{b}^1_j, \dots) = \left(\frac{2G(1)^2}{3}, \frac{G(1)}{6}, \frac{G(2)}{18}, \dots, \frac{G(j-1)}{3^{j-1}2}, \dots\right),$$

$$(\hat{q}^2_2, \hat{b}^2_1, \hat{b}^2_3, \dots, \hat{b}^2_j, \dots) = \left(\frac{2G(1)^2}{9}, \frac{G(1)}{2}, \frac{G(2)}{18}, \dots, \frac{G(j-1)}{3^{j-1}2}, \dots\right),$$

$$(\hat{q}^3_3, \hat{b}^3_1, \hat{b}^3_2, \hat{b}^3_4, \dots, \hat{b}^3_j, \dots) = \left(\frac{2G(1)G(2)}{27}, 0, 0, \frac{G(3)}{54}, \dots, \frac{G(j-1)}{3^{j-1}2}, \dots\right),$$

$$(\hat{q}^t_t, \hat{b}^t_1, \dots, \hat{b}^t_{t-1}, \hat{b}^t_{t+1}, \dots) = \left(\frac{2G(1)G(t-1)}{3^t}, 0, \dots, 0, \frac{G(t)}{3^t 2}, \dots\right),$$

with $G(y) = \left(1 - \frac{1}{yk}\right)$, for $t = 4, 5, \dots$.

PROOF: Assumptions 1 and 7 are satisfied by construction. As the initial endowments are the same of Example 1, Assumptions 2 and 3 are satisfied. Each utility function is additively separable in each commodity, linear in commodity money and the other commodity in the trader's initial endowment, and have logarithm utility for all other commodities. The discount factor 3^{-j} guarantees continuity in the product topology. Then, Assumption 4 is satisfied.[13] Since $v^t(x_0, x_t) = \frac{2}{3}x_0 + \frac{1}{2}x_t$, for $t = 1, 2, \ldots,$ $z^2(x_1, \ldots, x_{t-1}, x_{t+1} \ldots) = 3^{-1}\ln x_1 + \sum_{j=3}^{\infty} 3^{-j}\ln x_j$, and $z^t(x_1, \ldots, x_{t-1}, x_{t+1} \ldots) = \sum_{j=t+1}^{\infty} 3^{-j}\ln x_j$, for each $t = 1, 3, 4, \ldots,$ Assumption 5(i) is satisfied. Moreover, as the function $v^t(x_0, x_t)$ is the same of Example 1, for $t = 1, 2, \ldots,$ it follows that Assumptions 5(ii) and 5(iii) are satisfied. Furthermore, the traders of type 1 and 2 satisfy Assumption 6. In fact, their marginal utilities of commodity money are bounded, as they are equal to $\frac{2}{3}$ for any commodity bundle, and $z_j^t(x_j) = \ln x_j$ is such that $\lim_{x_j \to 0} \frac{\partial z_j^t}{\partial x_j}(x_j) = \infty$, for each $j \neq 0, t$, for $t = 1, 2$. Finally, it is straightforward, though laborious, to verify that the strategies at the Cournot-Nash equilibrium satisfy the first order conditions associated to the payoff maximisation problem (5), for $t = 1, 2, \ldots.$ *Q.E.D.*

4. EXISTENCE OF A WALRAS EQUILIBRIUM

In this section, we show the existence of a Walras equilibrium for an exchange economy satisfying Assumptions 1–4. The very first existence result of a Walras equilibrium in exchange economies with an infinity of commodities was proved by Bewley (1972). Subsequently, Wilson (1981) generalised the existence result to exchange economies with a double infinity of commodities and traders. Our existence theorem is based on this latter paper since our exchange economy is a particular case of the one it considers. In fact, his assumptions on consumption sets, initial endowments, and preferences are more general than ours. Only the assumption that the economy is irreducible does not have an explicit counterpart in our paper. However, the fact that all utility functions are strongly monotone in commodity money and that all traders hold commodity money implies irreducibility.

We first introduce some additional notation and definitions from Wilson (1981). Let X^i be trader i's consumption set and $\succsim^i$ trader i's preference relation over X^i. We define an exchange economy $\mathcal{E}$ as a set $\{X^i, \succsim^i, w^i\}_{i \in I}$. For each $x \in X^i$, let $P^i(x) = \{z \in X^i : z \succ^i x\}$ and $P_{-1}^i(x) = \{z \in X^i : x \succ^i z\}$. In words, $P^i(x)$ represents the set of commodity bundles which trader i strictly prefers to bundle x and $P_{-1}^i(x)$ represents those bundles to which x is strictly preferred. An economy is irreducible if, for any partition of I into two non-empty subset H_1 and H_2 and any allocation $\mathbf{x}$, there is a trader $i \in H_1$ and a commodity bundle y, with $y_j \leq \sum_{i \in H_2} w_j^i$ for each $j \in J$, such that $y + x^i \succ^i x^i$. We now state Wilson (1981)'s Theorem 2 with the assumptions considered in Section 6.3 of his paper.

THEOREM (Wilson's Existence Theorem) Consider an exchange economy $\mathcal{E} = \{X^i, \succsim^i, w^i\}_{i \in I}$. Suppose that

(I) for each $i \in I$: $X^i = \ell_\infty^+$ and the vector of aggregate initial endowments lies in ℓ_∞^+;

(II) for each $i \in I$ and $x \in X^i$: $P^i(x)$ and $P_{-1}^i(x)$ are both open relative to X^i with respect to the Mackey topology in ℓ_∞^+ and if $z \in P^i(x)$, then $\lambda z + (1-\lambda)x \in P^i(x)$ for $0 < \lambda \leq 1$;

(III) for each $i \in I$ and $x \in X^i$: if $z \in P^i(x)$ and $y \geq z$, then $y \in P^i(x)$;

(IV) I is a countable set;

(V) $\sum_{i \in I} w_j^i > 0$, for each $j \in J$;

[13]Logarithmic functions facilitate computations but are not defined at the boundary. This does not affect the current analysis but should be kept in mind.

(VI) the economy $\mathcal{E}$ is irreducible and, for any finite set of traders $H \subset I$, there is a finite set $G \subset I$ which contains H, for which the corresponding subeconomy $\{X^i, \succsim^i, w^i\}_{i \in G}$ is irreducible;

(VII) for each $i \in I$: $w^i_j > 0$ for only a finite number of j.

Then, there exists a Walras equilibrium for $\mathcal{E}$.

Proof: It is an immediate consequence of Theorem 2 in Wilson (1981). As he remarked in Section 6.3, the result of his Theorem 2 does not change if the commodity space of each traders is ℓ^+_∞, preferences are Mackey continuous, and the aggregate initial endowment lies in ℓ^+_∞. *Q.E.D.*

We now state and prove the theorem of existence of a Walras equilibrium in our exchange economy ${}_k\mathcal{E}$.

Theorem 2 Under Assumptions 1–4, there exists a Walras equilibrium for ${}_k\mathcal{E}$.

Proof: By Assumptions 1, $X^i = \ell^+_\infty$, for each $i \in I$. Since $w \in \ell^+_\infty$, by Assumption 2, the aggregate initial endowment kw also belongs to ℓ^+_∞. Hence, (I) is satisfied. Since utility functions are continuous in the product topology in ℓ^+_∞, by Assumption 4, they are also continuous in the Mackey topology in ℓ^+_∞ (see Bewley (1972), p. 531). Since the utility functions are concave, by Assumption 4, they are also explicitly quasi-concave (see Takayama (1974) p. 112). Hence, the preferences underlying the utility functions satisfy (II). Since the utility functions are monotone, by Assumption 4, the preferences underlying the utility functions satisfy (III). Since the set of traders I is a countable set, by construction, (IV) is satisfied. As $kw_j > 0$, by Assumption 3, (V) is satisfied. We now show that (VI) is satisfied. We first note that all traders hold commodity money, by Assumption 3, and all utilities are strongly monotone with respect to commodity money, by Assumption 4. Then, it is immediate to see that for any partition of the set of traders I into two non-empty subset H_1 and H_2 and any allocation $\mathbf{x}$, there is a trader $i \in H_1$ and a commodity bundle y with $0 < y_0 \leq \sum_{i \in H_2} w^i_0$ and $y_j = 0$, for each $j \in J \setminus \{0\}$, such that $y + x^i \succ^i x^i$. But then, the economy $\mathcal{E}$ is irreducible. By following the same steps, it is possible to show that any subeconomy $\{X, u^i(\cdot), w^i\}_{i \in G}$ is irreducible with $G \subset I$. Hence, (VI) is satisfied. Finally, each trader i of type t holds only the commodities 0 and t, by Assumption 3, and then (VII) is satisfied. Hence, conditions (I)–(VII) of Theorem 2 are satisfied and then there exists a Walras equilibrium for the exchange economy ${}_k\mathcal{E}$. *Q.E.D.*

5. CONVERGENCE TO THE WALRAS EQUILIBRIUM

As a Walras equilibrium exists in our framework, we can now state and prove the classical convergence result to it. More specifically, the next theorem shows that if the number of traders of each type tends to infinity then the price vector and the allocation, at an interior active type-symmetric Cournot-Nash equilibrium, converge to the Walras equilibrium of the underlying exchange economy. Before to formally state this theorem, we need to introduce some further notation and definitions.

As we need to consider a sequence of Cournot-Nash equilibria for strategic market games with different k, we write ${}_k\hat{s}$ to denote a type-symmetric Cournot-Nash equilibrium of the game ${}_k\Gamma$.[14] For each type-symmetric Cournot-Nash equilibrium ${}_k\hat{s}$ we denote by ${}_k\tilde{s}$ a vector whose elements are the trader types strategies, i.e., ${}_k\tilde{s} \in \prod_{t=1}^{\infty} S^t$ and ${}_k\tilde{s}^t = {}_k\hat{s}^t$, for

[14] To avoid cumbersome notation, in the previous part of the paper we have not written ${}_k\hat{s}$ even if the Cournot-Nash equilibrium belongs to a game ${}_k\Gamma$.

$t = 1, 2, \ldots$. Furthermore, we denote by ${}_k\tilde{p}$ a price vector such that ${}_k\tilde{p}_j = p_j({}_k\hat{s})$, for each $j \in J \setminus \{0\}$. Finally, an *interior active type-symmetric Cournot-Nash equilibrium* is an active type-symmetric Cournot-Nash equilibrium such that $\sum_{j \neq 0,t} {}_k\hat{b}_j^t < w_0^t$, for $t = 1, 2, \ldots$.

Theorem 3 Consider a sequence of games $\{{}_k\Gamma\}_k$. Suppose that there exists a sequence of interior type-symmetric active Cournot-Nash equilibria, $\{{}_k\hat{s}\}_k$, such that the sequences $\{{}_k\tilde{s}\}_k$ and $\{{}_k\tilde{p}\}_k$ converge to $\tilde{s}$ and to $\tilde{p}$ respectively. Let ${}_h\tilde{\mathbf{x}}$ be an allocation such that the commodity bundle of the h traders of type t is $x^t(\tilde{s})$, for $t = 1, 2, \ldots$. Then, the pair $((1, \tilde{p})), {}_h\tilde{\mathbf{x}})$ is a Walras equilibrium of the exchange economy ${}_h\mathcal{E}$, for any h.[15]

It is worth noting that Theorem 3 applies to the exchange economy in Example 2 as each active type-symmetric Cournot-Nash equilibrium belonging to the sequence $\{{}_k\hat{s}\}_k$ is interior and $\{{}_k\tilde{s}\}_k$ and $\{{}_k\tilde{p}\}_k$ converge to $\tilde{s}$ and to $\tilde{p}$ respectively.

To prove the convergence theorem we first prove a lemma which shows that a strategy ${}_k\hat{s}^i$ is also the solution of a maximisation problem where traders choose their best strategies as if they have no influence on the price vector which is fixed to ${}_kp^i$. This new maximisation problem is a key element of the convergent result. Given an active type-symmetric Cournot-Nash equilibrium ${}_k\hat{s}$, the vector ${}_kp^i$ is defined as follows

$$ {}_kp_t^i = p_t({}_k\hat{s})\left(1 - \frac{{}_k\hat{q}_t^i}{{}_k\bar{\hat{q}}_t}\right) \text{ and } {}_kp_j^i = p_j({}_k\hat{s})\left(1 + \frac{{}_k\hat{b}_j^i}{{}_k\bar{\hat{b}}_j^i}\right), \text{ for each } j \in J \setminus \{0, t\}. \tag{15} $$

Note that ${}_kp^i$ is a positive vector as the Cournot-Nash equilibrium ${}_k\hat{s}$ is active. Given the vector ${}_kp^i$ and a strategy $s^i \in S^i$, the commodity bundle $x^i(s^i, {}_kp^i)$ of a trader i of type t is such that[16]

$$ \begin{aligned} x_0^i(s^i, {}_kp^i) &= w_0^i - \sum_{j \neq 0,t} b_j^i + q_t^i {}_kp_t^i, \\ x_t^i(s^i, {}_kp^i) &= w_t^i - q_t^i, \\ x_j^i(s^i, {}_kp^i) &= \frac{b_j^i}{{}_kp_j^i}, \text{ for each } j \in J \setminus \{0, t\}. \end{aligned} \tag{16} $$

When prices are fixed at ${}_kp^i$, the objective function of a trader i becomes $u^i(x^i(s^i, {}_kp^i))$, for each $i \in I$. We now prove the analogue of Lemma 4 of Dubey and Shubik (1978) for a setting with infinitely many commodities.

Lemma 4 Given an interior active type-symmetric Cournot-Nash equilibrium ${}_k\hat{s}$ of the game ${}_k\Gamma$, the strategy ${}_k\hat{s}^i$ of a trader i of type t solves the following maximisation problem

$$ \begin{array}{lll} \max\limits_{s^i} & u^i(x^i(s^i, {}_kp^i)), & \\ \text{subject to} & q_t^i \leq w_t^i, & (i) \\ & \sum\limits_{j \neq 0,t} b_j^i \leq w_0^i, & (ii) \\ & -q_t^i \leq 0, & (iii) \\ & -b_j^i \leq 0, \text{ for each } j \in J \setminus \{0, t\}, & (iv) \end{array} \tag{17} $$

for each $i \in I$, for $k \geq 2$.

[15] The price vector $\tilde{p}$ does not include the price of commodity money. The first element of the price vector $(1, \tilde{p})$ is the price of commodity money.

[16] In order to save in notation, with some abuse, we denote by $x^i(\cdot)$ both the function $x^i(s)$ and the function $x^i(s^i, {}_kp^i)$.

Proof: Let ${}_k\hat{s}$ be an interior active type-symmetric Cournot-Nash equilibrium of the game ${}_k\Gamma$. Consider, without loss of generality, a trader i of type t. Following the same steps used in the proof of Lemma 2, it is possible to show that any $s^i \in S^i$ is a regular point of the constrained set. Since the utility function $u^i(\cdot)$ is Fréchet differentiable, by Assumption 4, and ${}_kp^i_j > 0$, for each $j \in J \setminus \{0\}$, as ${}_k\hat{s}$ is an active Cournot-Nash equilibrium, we have that trader i's objective function $u^i(x^i(s^i, {}_kp^i))$ is Fréchet differentiable because it is a composition of Fréchet differentiable functions. Therefore, it is immediate to see that all the hypothesis of the Generalised Kuhn-Tucker Theorem are satisfied and then, if a strategy s^i solves the maximization problem, there exist non-negative multipliers λ^i_1, λ^i_2 and μ^i_j, for each $j \in J \setminus \{0\}$, such that

$$\begin{aligned}
&\frac{\partial u^i}{\partial x_0}(x^i(s^i, {}_kp^i))\,{}_kp^i_t - \frac{\partial u^i}{\partial x_t}(x^i(s^i, {}_kp^i)) - \lambda^i_1 + \mu^i_t = 0, \\
&\lambda^i_1(q^i_t - w^i_t) = 0, \\
&\mu^i_t q^i_t = 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&-\frac{\partial u^i}{\partial x_0}(x^i(s^i, {}_kp^i)) + \frac{\partial u^i}{\partial x_j}(x^i(s^i, {}_kp^i))\frac{1}{{}_kp^i_j} - \lambda^i_2 + \mu^i_j = 0, \text{ for each } j \in J \setminus \{0, t\} \\
&\lambda^i_2\Big(\sum_{j\neq 0,t} b^i_j - w^i_0\Big) = 0, \\
&\mu^i_j b^i_j = 0, \text{ for each } j \in J \setminus \{0, t\}.
\end{aligned} \tag{19}$$

By the definition of ${}_kp^i$, it is immediate to verify that equations (8) and (11) become (18) and (19) respectively. Then, as ${}_k\hat{s}^i$, ${}_k\hat{\lambda}^i_1$, ${}_k\hat{\lambda}^i_2$, and ${}_k\hat{\mu}^i_j$, for each $j \in J \setminus \{0\}$, satisfy (8) and (11), they also satisfy the first order conditions associated to the maximization problem (17). Since $u(\cdot)$ is concave, by Assumption 4, and the price vector ${}_kp^i$ is fixed, it follows that $u^i(x^i(s^i, {}_kp^i))$ is a concave function. But then, ${}_k\hat{s}^i$ is optimal for the problem (17), for each $i \in I$, for $k \geq 2$.[17] *Q.E.D.*

We now prove Theorem 3.

Proof of Theorem 3: Consider a sequence of games $\{{}_k\Gamma\}_k$. Assume that there exists a sequence of interior active type-symmetric Cournot-Nash equilibria $\{{}_k\hat{s}\}_k$ such that the sequences $\{{}_k\tilde{s}\}_k$ and $\{{}_k\tilde{p}\}_k$ converge to $\tilde{s}$ and to $\tilde{p}$ respectively. Consider, without loss of generality, a trader i of type t. We first prove that the commodity bundle $x^i({}_k\hat{s}^i, {}_kp^i)$ maximises the utility function $u^i(\cdot)$ in the budget set $B^i(1, {}_kp^i)$.[18] First, we show that $x^i({}_k\hat{s}^i, {}_kp^i)$ belongs to $B^i(1, {}_kp^i)$. By the equations in (16) and the definition of the budget set, we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} {}_kp^i_j x^i_j({}_k\hat{s}^i, {}_kp^i) &= 1\Bigg(w^i_0 - \sum_{j\neq 0,t} {}_k\hat{b}^i_j + {}_k\hat{q}^i_t\,{}_kp^i_t\Bigg) + {}_kp^i_t(w^i_t - {}_k\hat{q}^i_t) + \sum_{j\neq 0,t} {}_kp^i_j \frac{{}_k\hat{b}^i_j}{{}_kp^i_j} \\
&= w^i_0 + {}_kp^i_t w^i_t
\end{aligned}$$

But then, $x^i({}_k\hat{s}^i, {}_kp^i) \in B^i(1, {}_kp^i)$. We now proceed by contradiction and we suppose that $x^i({}_k\hat{s}^i, {}_kp^i)$ is not a maximum point in the budget set. Then, there exists a commodity bundle $x'^i \in B^i(1, {}_kp^i)$ in the neighbourhood of $x^i({}_k\hat{s}^i, {}_kp^i)$ such that $u^i(x'^i) > u^i(x^i({}_k\hat{s}^i, {}_kp^i))$,

[17] This conclusion can be also obtained by Theorem 2 of Section 8.5 and Lemma 1 of Section 8.7 in Luenberger (1969).

[18] The price vector ${}_kp^i$ does not include the price of commodity money. The first element of the price vector $(1, {}_kp^i)$ is the price of commodity money.

as $u^i(\cdot)$ is concave by Assumption 4. Since $u^i(\cdot)$ is also monotone, it follows that $x'^i_j > x^i_j({}_k\hat{s}^i, {}_kp^i)$, for at least one commodity $j \in J$. As ${}_k\hat{s}^i$ is an interior active type-symmetric Cournot-Nash equilibrium, we have that $\sum_{j\neq 0,t} {}_k\hat{b}^i_j < w^i_0$ and $-{}_k\hat{q}^i_t < 0$. But then, as prices are fixed, there exists a feasible strategy $s'^i \in S^i$ such that $x^i_j(s'^i, {}_kp^i) = x'^i_j$. Hence, $u^i(x^i(s'^i, {}_kp^i)) > u^i(x^i({}_k\hat{s}^i, {}_kp^i))$. But this contradicts the fact that ${}_k\hat{s}^i$ solves the maximization problem (17), by Lemma 4. Therefore, the commodity bundle $x^i({}_k\hat{s}^i, {}_kp^i)$ maximises $u(\cdot)$ on $B^i(1, {}_kp^i)$, for each $i \in I$, for $k \geq 2$. In the next step of the proof we show that $\lim_{k\to\infty} x^i({}_k\tilde{s}^i, {}_kp^i)$ maximises the utility function on the budget set. Consider, without loss of generality, a trader of type t. We first note that $x^t({}_k\hat{s}^t, {}_kp^t) = x^t({}_k\tilde{s}^t, {}_kp^t)$, by the definition of ${}_k\tilde{s}^t$. Moreover, $\lim_{k\to\infty} {}_k\tilde{s}^t = \tilde{s}^t$, for $t = 1, 2, \ldots$, by the assumptions of the theorem. Since $p_j({}_k\hat{s}) \in [C, D]$, for each $j \in J \setminus \{0\}$, for any $k \geq 2$, by Lemma 3, and ${}_k\tilde{p} = p({}_k\hat{s})$, by definition, we have that ${}_k\tilde{p}_j \in [C, D]$, for each $j \in J \setminus \{0\}$, for any $k \geq 2$. Then, $\tilde{p} = \lim_{k\to\infty} {}_k\tilde{p}$ is such that $\tilde{p}_j \in [C, D]$, for each $j \in J \setminus \{0\}$, as the product topology is the topology of coordinate-wise convergence. Additionally, $\lim_{k\to\infty} {}_kp^t = \tilde{p}$ as in equations (15) the terms in brackets converges to 1 and we can substitute $p({}_k\hat{s})$ with ${}_k\tilde{p}$. These results imply that $\lim_{k\to\infty} x^t({}_k\tilde{s}^t, {}_kp^t) = x^t(\tilde{s}^t, \tilde{p})$, as the function defined by the equations in (16) is continuous at $\tilde{s}^t$ and $\tilde{p}$, where $\tilde{p}_j \in [C, D]$, for each $j \in J \setminus \{0\}$. Next, it is immediate to see that $x^t(\tilde{s}^t, \tilde{p}) = x^t(\tilde{s})$, by equations (1)–(3) and (16), and that $x^t(\tilde{s})$ belongs to $B^t(1, \tilde{p})$. As the commodity bundle $x^t(\tilde{s})$ is a point of continuity for $u^t(\cdot)$, we can conclude that $x^t(\tilde{s})$ maximises the utility function on the budget set $B^t(1, \tilde{p})$, for $t = 1, 2, \ldots$. Finally, the price formation rule guarantees that ${}_h\tilde{\mathbf{x}}$ is an allocation, for any h. Hence, $((1, \tilde{p}), {}_h\tilde{\mathbf{x}})$ is a Walras equilibrium for the exchange economy ${}_h\mathcal{E}$, for any h. *Q.E.D.*

6. CONCLUSION

In this paper, we have extended the analysis of noncooperative oligopoly to exchange economies with a countable infinite number of commodities and trader types. We have done so by considering the strategic market game, with commodity money and trading posts, analysed by Dubey and Shubik (1978). We have restricted our model to a multilateral oligopoly setting that was previously studied in Shubik (1973). For this game, we have proved the existence of an active Cournot-Nash equilibrium and its convergence to the Walras equilibrium when the number of traders of each type tends to infinity.

Our contribution differs from the one in Dubey and Shubik (1978) because we have proved the existence of a Cournot-Nash equilibrium where all commodities are exchanged (an active Cournot-Nash equilibrium) while they proved the existence of a Cournot-Nash equilibrium having positive prices but in which some commodities may not be exchanged (an equilibrium point). It is easy to see, in the proof of Theorem 1, that only Assumptions 1–5 are required to prove the existence of an equilibrium point in our model. As Assumptions 1–4 are comparable to the ones made by Dubey and Shubik (1978), it is the strong Assumption 5 that characterises the analysis of strategic market games in infinite economies. Such assumption is key to prove that prices have uniform lower and upper bounds without using the Uniform Monotonicity Lemma. Furthermore, Assumption 6 is not peculiar to our model as it is also needed in strategic market games with a finite set of commodities to prove the existence of a Cournot-Nash equilibrium where all commodities are exchanged. It is easy to see that the proof of Theorem 1 can be adapted to show the existence of an active Cournot-Nash equilibrium in the multilateral oligopoly model considered by Shubik (1973). The existence of an active Cournot-Nash equilibrium in the Dubey and Shubik (1978)'s contribution remains an open problem.

From an economic point of view, it is interesting to note that in our framework the relationship between traders' market shares and traders' market power is unclear. Given the price formation rule in the strategic market game and since the set of traders is countable, all traders will have market power on all commodities, i.e., all traders act strategically. However, the market share of all traders in Example 2 converges to zero along the sequence of trading post.[19] This phenomenon queries the appropriateness of using traders' market share in assessing the level of competition in trading posts. We plan to further study this issue in future research.

MATHEMATICAL APPENDIX

In this appendix, we describe the mathematical notions that we have used in the paper. The definitions and the theorem are based on Luenberger (1969) and the page number in brackets refers to it.

Definition (ℓ_∞ spaces) The space ℓ_∞^+ consists of non-negative bounded sequences (p. 29).

Definition (Fréchet differentiable) Let $f(\cdot)$ be a function defined on an open domain E in a normed space X and having range in a normed space Y. If for fixed $x \in E$ and each $h \in X$ there exists $f'(x)h \in Y$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|} = 0,$$

then $f(\cdot)$ is said to be Fréchet differentiable at x, $f'(x)h$ is said to be the Fréchet differential of $f(\cdot)$ at x with incremental h, and $f'(\cdot)$ is said to be Frechet derivative of $f(\cdot)$ (p. 172).

Definition (Continuously Fréchet differentiable) If $f'(\cdot)$ is continuous at the point x_0, we say that the Frechet derivative of $f(\cdot)$ is continuous at x_0. If the derivative of $f(\cdot)$ is continuous on some open sphere E, we say that $f(\cdot)$ is continuously Fréchet differentiable on E (p. 175).

Luenberger states the Regular Point definition (p. 248) and the Generalised Kuhn-Tucker Theorem (p. 249-250) for vector spaces. Since we deal with normed spaces, we state them for these particular spaces (see Example 1, p.250).

Definition (Regular Point) Let X be a normed vector space and let Z be a normed vector space with a closed positive cone having non-empty interior. Let $g(\cdot)$ be a function $g : X \to Z$ which is Fréchet differentiable. A point $x^* \in X$ is said to be a regular point of the inequality $g(x) \leq 0$ if $g(x^*) \leq 0$ and there is an $h \in X$ such that $g(x^*) + g'(x^*) \cdot h < 0$.

Theorem (Generalised Kuhn-Tucker Theorem) Let X be a normed vector space and let Z be a normed vector space with a closed positive cone having non-empty interior. Let $f(\cdot)$ be a Fréchet differentiable real-valued function on X and $g(\cdot)$ a Fréchet differentiable mapping from X into Z. Suppose x^* maximises $f(\cdot)$ subject to $g(x) \leq 0$ and that x^* is a regular point of the inequality $g(x) \leq 0$. Then there is a $z^* \geq 0$ such that

$$f'(x^*) + z^* g'(x^*) = 0,$$
$$z^* \cdot g(x^*) = 0.$$

[19] At the active type-symmetric Cournot-Nash equilibrium of Example 2 there are $j-1$ trader types active in a trading post for commodity $j \geq 2$ and all of them make the same bid. Then, the market share of a type t trader on commodity j, $\hat{b}_j^t / \bar{\hat{b}}_j$, is equal to $\frac{1}{(j-1)k}$, for $j \geq 2$. Hence, $\lim_{j \to \infty} \hat{b}_j^t / \bar{\hat{b}}_j = 0$, for $t = 1, 2, \ldots$.

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